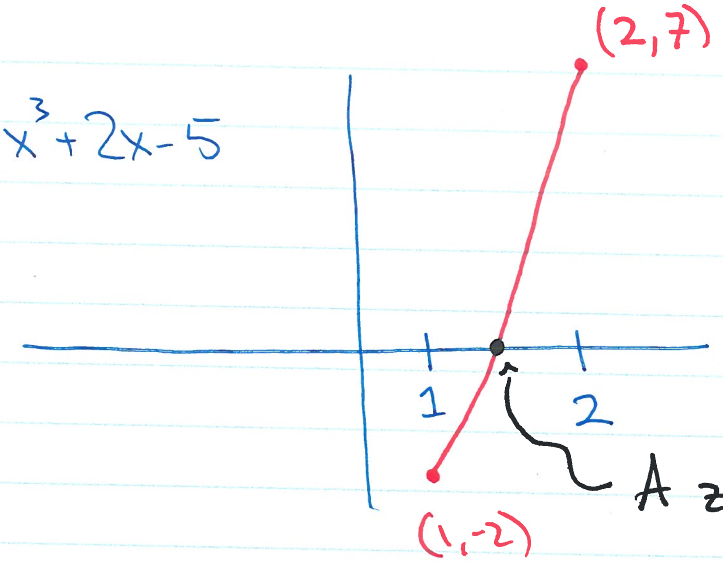
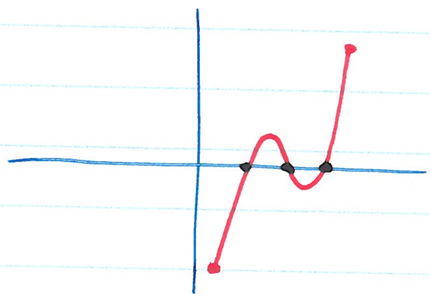
(a) $f(x)=x^3+2x-5$



$f(1)=1^3+2-5=-2$

$f(2)=2^3+2\cdot 2-5$
$=8+4-5=7$

(b) Why can we be sure it doesn't look like this?

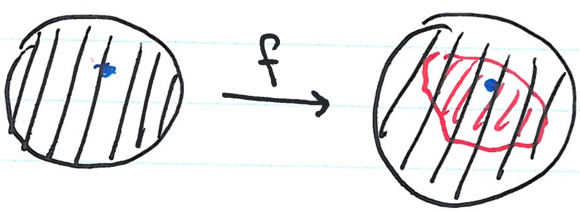


Just for fun: The IVT ~~can be~~ is equivalent to:

"For any continuous function $g:[a,b]\longrightarrow[a,b]$, there is some $x\in[a,b]$ such that $g(x)=x$."

(Take $f(x)=g(x)-x$ gives the original IVT.)

Two-dimensional version:
(Brouwer fixed point theorem)



"Any continuous function from a disk to itself has a fixed point."

MATH 220.204, MARCH 13 2019

(1) **The Intermediate Value Theorem:** Let a, b be real numbers such that $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0$ and $f(b) > 0$. There there is some real number x such that $a < x < b$ and $f(x) = 0$.

(a) Use the Intermediate Value Theorem to prove that the function $x^3 + 2x - 5 = 0$ has a solution on the interval $[1, 2]$.

Consider the function $f : [1,2] \longrightarrow \mathbb{R}$.

Then $f(1) = -2 < 0$, $f(2) = 7 > 0$. Therefore, by IVT, there exists some $x \in [1,2]$ such that $f(x) = 0$.

We are trying to prove that if $c \neq d$ then can't have $f(c) = f(d) = 0$. (Injectivity)

(b) Use proof by contradiction to prove that $x^3 + 2x - 5 = 0$ has **exactly one solution** on the interval $[1, 2]$. (You do not need to use any calculus here!)

Suppose there are two roots c and d, ie $f(c) = f(d) = 0$

Say that $1 \le c < d \le 2$.

$$c^3 + 2c - 5 = d^3 + 2d - 5 = 0$$

$c^3 + 2c = d^3 + 2d$. But because $c < d$, we know $c^3 < d^3$ and $2c < 2d$. Contradiction! (Therefore, only one root.)

(2) **Convergence:** We say the sequence $a_1, a_2, a_3, \ldots$ *converges to* L if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$.

Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{a_n}$. Prove that the sequence $a_1, a_2, a_3, \ldots$ does not converge.

$a_1 = 1$, $a_{n+1} = f(a_n)$, $f(x) = x + \frac{1}{x}$

Proof 1: Suppose that $a_1, a_2, \ldots$ converges to a limit, L. (For a contradiction.)

Notice that $a_2 = 2$ and $a_n > 2$ for every $n \ge 3$. Thus, $L > 2$.

For every ε, $\exists N$, $\forall n > N$, $|a_n - L| < \varepsilon$.

So this should hold when $\varepsilon = \frac{1}{2L}$. That is, there is some $N \in \mathbb{N}$ s.t $\forall n > N$, $|a_n - L| < \frac{1}{2L}$. This implies that $L - \frac{1}{2L} < a_{N+1} < L + \frac{1}{2L}$.

Then $a_{N+2} = a_{N+1} + \frac{1}{a_{N+1}} > L - \frac{1}{2L} + \frac{1}{L + \frac{1}{2L}}$

$$= L - \frac{1}{2L} + \frac{2L}{2L^2+1} > L - \frac{1}{2L} + \frac{2L}{3L^2} = L - \frac{1}{2L} + \frac{2}{3L} = L + \frac{1}{6L}.$$

next page

But then it follows that $a_{N+2} > L + \frac{1}{6L}$. Thus, $\forall n > N+1$, $a_n > L + \frac{1}{6L}$.

It follows that if we pick $\varepsilon = \frac{1}{6L}$, it is NOT true that $|a_n - L| < \varepsilon$ for n large.

This is a contradiction! $\rightarrow\leftarrow$

Thus, no limit L exists. ∎

Proof 2 (by Xinpei): For every $n \geq 1$,

$$a_{n+1} = a_n + \frac{1}{a_n}$$

$$a_{n+1}^2 = a_n^2 + 2 + \frac{1}{a_n^2}$$

$$a_{n+1}^2 - a_n^2 = 2 + \frac{1}{a_n^2}$$

Therefore, $a_{n+1}^2 - a_1^2 = \sum_{k=1}^{n}(a_{k+1}^2 - a_k^2) = \sum_{k=1}^{n}\left(2 + \frac{1}{a_k^2}\right)$

$$= 2n + \sum_{k=1}^{n} \frac{1}{a_k^2} > 2n$$

So since $a_1 = 1$, we get $a_{n+1} > \sqrt{2n+1}$. So $a_n > \sqrt{2n-1}$

Now for every $L > 0$, we find that if $n \geq \frac{L^2+1}{2}$, then $a_n > L$. In particular, there is a positive number x such that $n \geq \frac{L^2+1}{2} \Rightarrow a_n \geq L + x$. So if we take $\varepsilon = x$, it is NOT true that $\exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \varepsilon$.

So the sequence $a_1, a_2, \ldots$ doesn't converge to L.

This holds for every L, so $a_1, a_2, \ldots$ doesn't converge. ∎